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N-Player Games
N-Player Games

Games with \( N \)-players \( P_1, P_2, \ldots, P_N \), allowed to select policies within action spaces \( \Gamma_1, \Gamma_2, \ldots, \Gamma_N \). When

\[
\begin{align*}
P_1 & \text{ uses policy } \gamma_1 \in \Gamma_1 \\
P_2 & \text{ uses policy } \gamma_2 \in \Gamma_2 \\
& \quad \vdots \\
P_N & \text{ uses policy } \gamma_N \in \Gamma_N
\end{align*}
\]

the outcome of the game for player \( P_i \) is denoted by

\[ J_i(\gamma_1, \gamma_2, \ldots, \gamma_N) \]

Each \( P_i \) wants to minimize their own outcome, and does not care about the outcome of the other players.
N-Player Games

To avoid writing all the policies, separate the dependence of $J_i$ on $\gamma_i$ and on the remaining policies $\gamma_{-i}$ and write

$$J_i(\gamma_i, \gamma_{-i})$$

with the abbreviation to denote a list of all but the $i$th policy

$$\gamma_{-i} \equiv (\gamma_1, \gamma_2, \cdots, \gamma_{i-1}, \gamma_{i+1}, \cdots, \gamma_N)$$

Terminology also applies to action spaces, as in

$$\gamma_{-i} \in \Gamma_{-i}$$

which is meant to be a short-hand notation for

$$\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2, \cdots, \gamma_{i-1} \in \Gamma_{i-1}, \gamma_{i+1} \in \Gamma_{i+1}, \cdots, \gamma_N \in \Gamma_N$$
Security Levels and Policies

Security policies for N-player games:

Finding the policy that guarantees the least possible cost, assuming the worse possible choice by the other players.

**Definition 11.1** (Security policy).

Security level for $P_i$, $i \in \{1, 2, \ldots, N\}$ is defined by

$$\bar{V}(J_i) := \inf_{\gamma_i \in \Gamma_i} \left( \sup_{\gamma_{-i} \in \Gamma_{-i}} I_i(\gamma_i, \gamma_{-i}) \right)$$

- minimize cost assuming worst choice by $P_i$
- worst choice by all other players $P_{-i}$ from $P_i$’s perspective
Security Levels and Policies

**Security policy** for $P_i$

Any policy $\gamma_i^* \in \Gamma_i$ for which the infimum is achieved, i.e.,

$$\bar{V}(J_i) := \inf_{\gamma_i \in \Gamma_i} \sup_{\gamma_{-i} \in \Gamma_{-i}} J_i(\gamma_i, \gamma_{-i}) = \sup_{\gamma_{-i} \in \Gamma_{-i}} J_i(\gamma_i^*, \gamma_{-i}^*)$$

$\gamma_i^*$ achieves the infimum

Security policies may not exist because the infimum may not be achieved by a policy in $\Gamma_i$.

An $N$-tuple of policies $(\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*)$ is said to be **minimax** if each $\gamma_i$ is a security policy for $P_i$.
Nash Equilibria

Definition 11.2 (Nash equilibrium). An $N$-tuple of policies

$$\gamma^* := (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*) \in \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_N$$

is a NE if

$$J_i(\gamma^*) = J_i(\gamma_i^*, \gamma_{-i}^*) \leq J_i(\gamma_i, \gamma_{-i}^*), \; \forall \gamma_i \in \Gamma_i, \; i \in \{1, 2, \ldots, N\}$$

and the $N$-tuple $(J_1(\gamma^*), J_2(\gamma^*), \ldots, J_N(\gamma^*))$ is called the Nash outcome of the game.

The NE is admissible if there is no better NE in the sense that there is no other

$$\bar{\gamma}^* := (\bar{\gamma}_1^*, \bar{\gamma}_2^*, \ldots, \bar{\gamma}_N^*) \in \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_N$$

such that

$$J_i(\bar{\gamma}^*) \leq J_i(\gamma^*), \; \forall i \in \{1, 2, \ldots, N\}$$

with a strict inequality for at least one player.
Pure N-Player Games in Normal Form
Pure N-Player Games in Normal Form

Played by $N$ players $P_1, P_2, \ldots, P_N$, each selecting policies from finite action spaces:

$P_i$ has available $m_i$ actions: $\Gamma_i := \{1, 2, \ldots, m_i\}$

Outcomes for $P_i$’s are quantified by $N$ tensors $A^1, A^2, \ldots, A^N$, each $N$-dimensional with dimensions $m_1, m_2, \ldots, m_N$. When

$$
\begin{align*}
P_1 & \text{ selects action } k_1 \in \Gamma_1 := \{1, 2, \ldots, m_1\} \\
P_2 & \text{ selects action } k_2 \in \Gamma_2 := \{1, 2, \ldots, m_2\} \\
& \quad \vdots \\
P_N & \text{ selects action } k_N \in \Gamma_N := \{1, 2, \ldots, m_N\}
\end{align*}
$$

the outcome for $P_i$ is obtained from the appropriate entry $a^i_{k_1 k_2 \ldots k_N}$ of the tensor $A^i$

- all players want to minimize their respective outcomes.
Pure N-Player Games in Normal Form

Testing if a particular $N$-tuple of pure policies $(k_1^*, k_2^*, \ldots, k_N^*)$ is a NE is straightforward. Just check if

$$a_{k_i^* k_{-i}^*}^i \leq a_{k_i k_{-i}}^i, \quad \forall k_i \in \{1, 2, \ldots, m_i\}, \forall i \in \{1, 2, \ldots, N\}$$

Finding a NE in pure policies is computationally difficult

- need to check all possible $N$-tuples, which are as many as

$$m_1 \times m_2 \times \cdots \times m_N$$

**Tensor**: a multi-dimensional array that generalizes the concept of matrix for dimensions higher than two.
Mixed Policies for N-Player Games in Normal Form
A mixed policy for player $P_i$ is a set of numbers

$$y^i := (y^i_1, y^i_2, \ldots, y^i_{m_i}), \quad \sum_{k=1}^{m_i} y^i_k = 1 \quad y^i_k \geq 0, \quad \forall k \in \{1, 2, \ldots, m_i\}$$

$y^i_k$: probability that $P_i$ uses to select action $k \in \{1, 2, \ldots, m_i\}$.

Each mixed policy $y_i$ is an element of the action space $Y_i$, consisting of probability distributions over $m_i$ actions.

Random selections by $P_i$’s are statistically independently
- each $P_i$ tries to minimize their own expected outcome:

$$J_i = \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \cdots \sum_{k_N=1}^{m_N} y^1_{k_1} y^2_{k_2} \cdots y^N_{k_N} a_{k_1 k_2 \cdots k_N}$$

Outcomes when $P_1$ selects $k_1$ and $P_2$ selects $k_2$ and \ldots
Mixed Policies for N-Player Games in Normal Form

Definition 11.3 (Mixed Nash equilibrium).

An $N$-tuple of policies $(y_1^*, y_2^*, \ldots, y_N^*) \in \mathcal{Y}^1 \times \mathcal{Y}^2 \times \cdots \times \mathcal{Y}^N$ is a mixed Nash equilibrium (MNE) if

$$\sum_{k_1} \sum_{k_2} \cdots \sum_{k_N} \left( \prod_{j \neq i} y_{k_j}^* \right) a_{k_1 k_2 \cdots k_N} \leq \sum_{k_1} \sum_{k_2} \cdots \sum_{k_N} \left( \prod_{j \neq i} y_{k_j}^* \right) a_{k_1 k_2 \cdots k_N},$$

or equivalently in a more compressed form

$$\sum_{k_1 k_2 \cdots k_N} y_{k_i}^* \left( \prod_{j \neq i} y_{k_j}^* \right) a_{k_1 k_2 \cdots k_N} \leq \sum_{k_1 k_2 \cdots k_N} y_{k_i} \left( \prod_{j \neq i} y_{k_j}^* \right) a_{k_1 k_2 \cdots k_N},$$

$$\forall i \in \{1, 2, \ldots, N\}.$$
(As in bimatrix games) The introduction of mixed policies enlarges the action spaces for both players to the point that NE now always exist.

**Theorem 11.1** (Nash).

Every $N$-player game in normal form has at least one mixed Nash Equilibrium.
Completely Mixed Policies
Computing NE for $N$-player games in normal form is not easy.

- simpler for games that admit completely mixed equilibria

**Definition 11.4** (completely mixed Nash equilibria (MNE))

A MNE $(y_1^*, y_2^*, \ldots, y_N^*)$ is **completely mixed** or an **inner-point equilibrium** if all probabilities are strictly positive, i.e.,

$$y_1^* > 0, \quad y_2^* > 0, \quad \cdots, \quad y_N^* > 0,$$

All completely MNE can be found by solving an algebraic multi-linear system of equations.
Completely Mixed Policies

Lemma 11.1 (completely mixed Nash equilibria).
If \((y^1*, y^2*, \ldots, y^N*)\) is a completely MNE with outcomes \((p^1*, p^2*, \ldots, p^N*)\) then

\[
\sum_{k \neq i} \left( \prod_{j \neq i} y^j_{k_j} \right) a_{k_1 k_2 \cdots k_N} = p^i*, \quad \forall i \in \{1, 2, \ldots, N\}
\]

Conversely, any solution \((y^1*, \ldots, y^N*), (p^1*, \ldots, p^N*)\) for which

\[
\sum_{k_i=1}^{m_i} y^i_{k_i} = 1, \quad y^i* \geq 0, \quad \forall i \in \{1, 2, \ldots, N\}
\]

corresponds to a MNE \((y^1*, y^2*, \ldots, y^N*)\) with outcomes \((p^1*, p^2*, \ldots, p^N*)\) for the original game, and for any similar game in which some/all players want to maximize instead of minimize their outcomes.
Completely Mixed Policies

Proof of Lemma 11.1.
Assuming \((y^1*, y^2*, \ldots, y^N*)\) is a completely MNE, we have

\[
\sum_{k_1, k_2, \ldots, k_N} y_{k_i}^i \left( \prod_{j \neq i} y_{k_j}^j \right) a_{k_1, k_2, \ldots, k_N}^i = \min_{y_i} \sum_{k_1, k_2, \ldots, k_N} y_{k_i}^i \left( \prod_{j \neq i} y_{k_j}^j \right) a_{k_1, k_2, \ldots, k_N}^i
\]

\[
= \min_{y_i} \sum_{k_i} y_{k_i}^i \sum_{k_{-i}} \left( \prod_{j \neq i} y_{k_j}^j \right) a_{k_1, k_2, \ldots, k_N}^i
\]

If one of the \(\sum_{k_{-i}} \left( \prod_{j \neq i} y_{k_j}^j \right) a_{k_1, k_2, \ldots, k_N}^i\) was strictly larger than any of the remaining ones, then the minimum would be achieved with \(y_i = 0\) and the NE would not be completely mixed. Therefore to have a completely MNE, we must have

\[
\sum_{k_{-i}} \left( \prod_{j \neq i} y_{k_j}^j \right) a_{k_1, k_2, \ldots, k_N}^i = p^i
\]
Completely Mixed Policies

Conversely, if \((y_1^*, y_2^*, \ldots, y_N^*)\) and \((p_1^*, p_2^*, \ldots, p_N^*)\) satisfy the two conditions in Lemma 11.1, then

\[
\sum_{k_1 k_2 \cdots k_N} y_{k_i}^* \left( \prod_{j \neq i} y_{k_j}^* \right) a_{k_1 k_2 \cdots k_N}^i = \sum_{k_1 k_2 \cdots k_N} y_{k_i}^* \left( \prod_{j \neq i} y_{k_j}^* \right) a_{k_1 k_2 \cdots k_N}^i = \min_{y_i} \sum_{k_i} y_{k_i}^* p_i^* = p_i^*, \quad \forall y_i \in Y_i
\]

which shows that \((y_1^*, y_2^*, \ldots, y_N^*)\) is a MNE with outcome \((p_1^*, p_2^*, \ldots, p_N^*)\).

In fact, \((y_1^*, y_2^*, \ldots, y_N^*)\) is also a MNE for a different game in which some/all \(P_i\)'s want to maximize instead of minimize the outcome.
End of Lecture

11 - N-Player Games

Questions?