COSC-6590/GSCS-6390

Games: Theory and Applications

Lecture 10 - Computation of Nash Equilibria for Bimatrix Games

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Completely Mixed Nash Equilibria

Alternative version of the battle of the sexes game

\[ A = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \] \quad \text{\(P_1\) choices} \quad \text{\(P_2\) choices} \quad \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} \] \quad \text{\(P_1\) choices} \quad \text{\(P_2\) choices}

To find a mixed Nash Equilibria (MNE), compute vectors

\[ y^* := [y_1^* \quad 1 - y_1^*]' \quad y_1^* \in [0, 1] \]

\[ z^* := [z_1^* \quad 1 - z_1^*]' \quad z_1^* \in [0, 1] \]

for which

\[ y'^* A z^* = y_1^* (1 - 6z_1^*) + 4z_1^* - 1 \leq y_1 (1 - 6z_1^*) + 4z_1^* - 1, \quad \forall y_1 \in [0, 1] \]

\[ y'^* B z^* = z_1^* (2 - 6y_1^*) + 5y_1^* - 2 \leq z_1 (2 - 6y_1^*) + 5y_1^* - 2, \quad \forall z_1 \in [0, 1] \]

true if RHS of: eq. 1 independent of \(y_1\); eq. 2 independent of \(z_1\)
Completely Mixed Nash Equilibria

In particular, by making

\[
\begin{align*}
1 - 6z_1^* &= 0 \\
2 - 6y_1^* &= 0
\end{align*}
\]

\(\Leftrightarrow\)

\[
\begin{align*}
z_1^* &= \frac{1}{6} \\
y_1^* &= \frac{1}{3}
\end{align*}
\]

This leads to the following MNE

\[
(y^*, z^*) = \left( \begin{bmatrix} \frac{1}{3} & 2 \\ \frac{1}{3} & 3 \end{bmatrix}', \begin{bmatrix} 1 & 5 \\ 6 & 6 \end{bmatrix}' \right)
\]

\(P_1\) (husband) goes to football 66% of times

\(P_2\) (wife) goes to football 83% of times

and outcomes

\[
(y^{'*}A z^*, y^{'*}B z^*) = (4z_1^* - 1, 5y_1^* - 2) = \left( -\frac{1}{3}, -\frac{1}{3} \right)
\]
This particular NE has the very special property that

\[ y^* \, A z^* = y' \, A z^*, \quad \forall y \in \mathcal{Y} \]
\[ y^* \, B z^* = y' \, B z, \quad \forall z \in \mathcal{Z} \]

then it is also a NE for a bimatrix game defined by the matrices \((-A, -B)\), i.e., exactly opposite objectives by both players.
Completely Mixed Nash Equilibria

Definition 10.1 (Completely Mixed Nash Equilibria (MNE)).

A MNE \((y^*, z^*)\) is completely mixed or an inner-point equilibria if all probabilities are strictly positive, i.e., \(y_i^*, z_j^* \in (0, 1), \forall i, j\).

Lemma 10.1 (Completely mixed NE).

If \((y^*, z^*)\) is a completely MNE with outcomes \((p^*, q^*)\) for a bimatrix game defined by the matrices \((A, B)\), then

\[ Az^* = p^* 1_{m \times 1}, \quad B'y^* = q^* 1_{n \times 1}, \]

Consequently, \((y^*, z^*)\) is also a MNE for the three bimatrix games defined by \((-A, -B), (A, -B)\) and \((-A, B)\).
Proof of Lemma 10.1.
Assuming that \((y^*, z^*)\) is a completely MNE for the game defined by \((A, B)\), we have that

\[
y^*'A z^* = \min_y y'Az^* = \min_y \sum_i y_i (A z^*)_i
\]

if row \(i\) of \(Az^*\) was strictly larger than any of the others, then the minimum would be achieved with \(y_i = 0\) and the NE would not be completely mixed. To have a completely MNE, we need all the rows of \(Az^*\) exactly equal to each other:

\[
Az^* = p^* 1_{m \times 1}
\]

for some scalar \(p^*\), which means that

\[
y^*'Az^* = y'Az^* = p^*, \quad \forall y, y^* \in \mathcal{Y}
\]
Completely Mixed Nash Equilibria

Similarly, since none of the $z_j = 0$, then all columns of $y^{'B}$ (the rows of $B'y^*$) must be equal to some constant $q^*$ and then

$$y^{'B}z^* = y^{'B}z = q^*, \quad \forall z, z^* \in Z$$

Therefore, we conclude $(p^*, q^*)$ is indeed the Nash outcome of the game and that $y^*, z^*$ is also a MNE for the three bimatrix games defined by $(-A, -B)$, $(A, -B)$ and $(-A, B)$.

**Corollary 10.1.** If $(y^*, z^*)$ is a completely mixed SPE for the zero-sum matrix game defined by $A$ with mixed value $p^*$, then

$$Az^* = p^*1_{m \times 1} \quad A'y^* = p^*1_{n \times 1}$$

for some scalar $p^*$. Consequently, $(y^*, z^*)$ is also a mixed SPE for the zero-sum matrix game $-A$, and a MNE for the two (non-zero-sum) bimatrix games $(A, A)$ and $(-A, -A)$. 

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Computation of Completely Mixed NE
Computation of Completely Mixed Nash Equilibria

Simple: all the equilibria must satisfy (see Lemma 10.1)

\[ B' y^* = q^* 1_{n \times 1} \quad 1'y^* = 1 \quad A z^* = p^* 1_{m \times 1} \quad 1' z^* = 1 \]

a linear system, with \( n + m + 2 \) equations and unknowns:

- \( m \) entries of \( y^* \), \( n \) entries of \( z^* \), and two scalars \( p^* \) and \( q^* \).

After solving, verify that the resulting \( y^* \) and \( z^* \) do have non-zero entries so that they belong to the sets \( \mathcal{Y} \) and \( \mathcal{Z} \)

- if they do, we conclude that we have found a NE.

**Lemma 10.2.** Suppose that the vectors \((y^*, z^*)\) satisfy the previous conditions. If all entries of \( y^* \) and \( z^* \) are non-negative, then \((y^*, z^*)\) is a mixed NE.
Computation of Completely Mixed Nash Equilibria

**Example 10.1** (Battle of the sexes BoS).

For the BoS game introduced before, the conditions become

\[
A z^* = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \frac{z_1^*}{1-z_1^*} \end{bmatrix} = \begin{bmatrix} p^* \\ p^* \end{bmatrix} = \begin{bmatrix} \frac{p^*}{p^*} \end{bmatrix} \\
B^t y^* = \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} \frac{y_1^*}{1-y_1^*} \end{bmatrix} = \begin{bmatrix} q^* \\ q^* \end{bmatrix}
\]

which is equivalent to

\[
p^* = -2z_1^*, \quad 4z_1^* = 1 + p^*, \quad q^* = -y_1^*, \quad 5y_1^* = 2 + q^*
\]

\[
\Rightarrow z_1^* = \frac{1}{6}, \quad y_1^* = \frac{1}{3}, \quad p^* = q^* = \frac{1}{3}
\]

as we had previously concluded.

However, we now know that this is the unique completely MNE for this game.
Computation of Completely Mixed Nash Equilibria

For the BoS game in Example 9.3, the conditions become

\[
A z^* = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_1^* \\ 1 - z_1^* \end{bmatrix} = \begin{bmatrix} p^* \\ p^* \end{bmatrix} \quad B' y^* = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1^* \\ 1 - y_1^* \end{bmatrix} = \begin{bmatrix} q^* \\ q^* \end{bmatrix}
\]

which is equivalent to

\[-3z_1 + 1 = p^*, \quad z_1^* - 1 = p^*, \quad -4y_1^* + 3 = q^*, \quad 4y_1^* - 2 = q^*
\]

\[\Rightarrow z_1^* = \frac{1}{2}, \quad y_1^* = \frac{5}{8}, \quad p^* = -\frac{1}{2}, \quad q^* = \frac{1}{2}\]

**Note:** the completely MNE is not admissible: it is strictly worse for both players than the pure NE that we found before.

- (1, 1) was a pure NE with outcomes (−2, −1)
- (2, 2) was a pure NE with outcomes (−1, −2)
Numerical Computation of MNE
Numerical Computation of Mixed Nash Equilibria

A systematic numeric procedure to find MNE for a bimatrix game defined by two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ expressing the outcomes of players $P_1$ and $P_2$.

$P_1$ selects a row of $A/B$ and $P_2$ selects a column of $A/B$.

MNE can be found by solving a **quadratic program**
Theorem 10.1. The pair of policies \((y^*, z^*) \in \mathcal{Y} \times \mathcal{Z}\) is a MNE with outcome \((p^*, q^*)\) if and only if the tuple \((y^*, z^*, p^*, q^*)\) is a (global) solution to the following minimization:

\[
\begin{align*}
\text{minimize} & \quad y'(A + B)z - p - q \\
\text{subject to} & \quad Az \geq p1 \\
& \quad B'y \geq q1 \\
& \quad y \geq 0 \\
& \quad 1y = 1 \\
& \quad z \geq 0 \\
& \quad 1z = 1 \\
\end{align*}
\]

\[(y \in \mathcal{Y}) \quad (z \in \mathcal{Z})\]

optimization over \(m+n+2\) parameters \((y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_n, p, q)\)

This minimization always has a global minima at zero.
Numerical Computation of Mixed Nash Equilibria

For zero-sum games, $A = -B$ and minimization is a linear program that finds both policies in one shot

Observation: efficiency is reduced!
- solving two small problems is better than solving a large one.

Attention! Unless $A = -B$ (a zero-sum game), the quadratic criteria is indefinite because we can select $z$ to be any vector for which $(A + B)z \neq 0$ and then obtain

A positive value with
\[ y = (A + B)z, \quad p = q = 0 \quad \Rightarrow \quad y'(A + B)'(A + B)z - p - q = \| (A + B)z \|^2 > 0 \]

and a negative value with
\[ y = -(A + B)z, \quad p = q = 0 \Rightarrow y'(A + B)'(A + B)z - p - q = -\| (A + B)z \|^2 < 0 \]
Numerical Computation of Mixed Nash Equilibria

Here, the quadratic criteria in the minimization is not convex: numerical solvers can get caught in local minima.

- **Solution:** verify that solver found a global minimum. Easily done: we know that the global minimum is zero.

If solver gets caught in a local minimum

- restart it at a different (typically random) initial point.
Numerical Computation of Mixed Nash Equilibria

MATLAB® Hint 3 (Quadratic programs).

MATLAB®’s Optimization Toolbox command

\[
[x,\text{val}] = \text{quadprog}(H,c,A\text{in},b\text{in},A\text{eq},b\text{eq},\text{low},\text{high},x0)
\]

numerically solves quadratic programs of the form

\[
\begin{align*}
\text{minimum} & \quad \frac{1}{2}x' H x + c' x \\
\text{subject to} & \quad A\text{in} x \leq b\text{in} \\
& \quad A\text{eq} x = b\text{eq} \\
& \quad \text{low} \leq x \leq \text{high}
\end{align*}
\]

and returns the value \text{val} of the minimum and a vector \text{x} that achieves the minimum.

To avoid the corresponding inequality constraints, the vector

- \text{low}: can have some or all entries equal to \(-\text{Inf}\)
- \text{high}: can have some or all entries equal to \(+\text{Inf}\)
Numerical Computation of Mixed Nash Equilibria

Vector $x_0$: (optional) starting point for the optimization

- important when $H$ is indefinite since in this case the minimization is not convex and may have local minima

MATLAB® command `optimset`

- used to set optimization options for `quadprog`
- allows the selection of the optimization algorithm, which for the computation of NE, must support non-convex problems.

The following MATLAB® code can be used to find a MNE to the bimatrix game defined by $A$ and $B$, starting from a random initial condition $x_0$. 
Numerical Computation of Mixed Nash Equilibria

```matlab
[m,n] = size(A);
x0 = rand(n+m+2,1);

% y'( A + b )z - p - q
H = [zeros(m,m), A+B, zeros(m,2); A'+B', zeros(n,n+2); zeros(2, m+n+2)];
c = [zeros(m+n,1); -1; -1];

% Az >= p & B' y >= q
Ain = [zeros(m,m), -A, ones(m,1), zeros(m,1); -B', zeros(n,n+1), ones(n,1)];
bin = zeros(m+n,1);

% sum(y) = sum(z)=1
Aeq = [ones(1,m), zeros(1,n+2); zeros(1,m), ones(1,n),0 ,0];
beq = [1;1];

% y.i , z.i in [0,1]
low = [zeros(n+m,1); -inf;-inf];
high = [ones(n+m,1);+inf;+inf];

% solve quadratic program
options = optimset('TolFun',1e-8,'TolX',1e-8,'TolPCG',1e-8,'Algorithm','active-set');

[x,val,exitflag ] = quadprog(H,c,Ain,bin,Aeq,beq,low,high,x0,options)

y = x(1:m)
z = x(m+1:m+n)
p = x(m+n+1)
q = x(m+n+2)
```
**Proof of Theorem 10.1.** Justify two statements separately:

1. A MNE is always a global minimum
2. Any global minimum is a MNE

For 1, assume \((y^*, z^*)\) is a MNE with outcome \((p^*, q^*)\) and show that \((y^*, z^*, p^*, q^*)\) is a global minimum. We need to show:

1. Point \((y^*, z^*, p^*, q^*)\) satisfies all the constraints in the minimization. Indeed, since

\[
p^* = y'^*Az^* \leq y'Az^*, \quad \forall y \in \mathcal{Y}
\]

in particular for every integer \(i \in \{1, 2, \ldots, m\}\) if we pick

\[
y = [0 \cdots 0 \ 1 \ 0 \cdots 0]'
\]

1 at the \(i\)th position
we conclude that

\[
p^* \leq (Az^*)_i
\]

$i$th entry of $Az^*$

and therefore $p^*1 \preceq Az^*$. On the other hand, since

\[
q^* = y^\prime B z^* \leq y^\prime B z, \ \forall z \in Z
\]

we also conclude that $q^*1 \preceq y^\prime B$. This entry-wise inequality between row vectors can be converted into an entry-wise inequality between column vectors by transposition: $q^*1 \preceq B^\prime y^*$.

Remaining constraints on $y^*$ and $z^*$ hold since $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$.
Numerical Computation of Mixed Nash Equilibria

2. \((y^*, z^*, p^*, q^*)\) achieves the global minimum, which is zero. Note that since \(p^* = y^*A z^*\) and \(q^* = y^*B z^*\) we have

\[ y^* (A + B) z^* - p^* - q^* = 0 \]

It remains to show that no other vectors \(y\) and \(z\) that satisfy the constraints can lead to a value for the criteria lower than zero:

\[
\begin{align*}
A z & \geq p 1 \\
B' y & \geq q 1
\end{align*}
\Rightarrow
\begin{align*}
y' A z & \geq p \\
z' B' y & \geq q
\end{align*}
\Rightarrow
\begin{align*}
y'(A + B) z - p - q & \geq 0
\end{align*}

To prove the converse statement assume \((y^*, z^*, p^*, q^*)\) is a global minimum, and show that \((y^*, z^*)\) must be a MNE with outcome \((p^*, q^*)\).

From **Theorem 9.1**: there is at least one MNE and we have seen above that this leads to a global minimum equal to zero.
Numerical Computation of Mixed Nash Equilibria

Therefore any global minimum \((y^*, z^*, p^*, q^*)\) must satisfy:

\[ y^*(A + B)z^* - p^* - q^* = 0 \]

From RHS eqs., we conclude that

\[ y'Az^* \geq p^*, \quad \forall y \in \mathcal{Y} \]
\[ z'B'y^* \geq q^*, \quad \forall z \in \mathcal{Z} \]

The proof is completed as soon as we show that

\[ y^*Az^* = p^*, \quad z^*B'y^* = q^* \]

To achieve this, set \(y = y^*\) and \(z = z^*\), which leads to

\[ y'Az^* - p^* \geq 0, \quad z'B'y^* - q^* \geq 0 \]

However, because of LHS eq., the two numbers in the LHS of these ineq. must add up to zero, so they must be equal to zero.

\[ y^*Az^* = p^* \quad \text{and} \quad z^*B'y^* = q^* \]

hold, and this completes the proof.
Practice Exercises
Practice Exercises

10.1. Use MATLAB® to compute NE policies for

1. battle of the sexes with

\[
A = \begin{bmatrix}
-2 & 0 \\
3 & -1
\end{bmatrix}
\] \quad \text{P}_1 \text{ choices}

\[
P_2 \text{ choices}
\]

\[
B = \begin{bmatrix}
-1 & 3 \\
0 & -2
\end{bmatrix}
\] \quad \text{P}_1 \text{ choices}

\[
P_2 \text{ choices}
\]

2. prisoners’ dilemma with

\[
A = \begin{bmatrix}
2 & 30 \\
0 & 8
\end{bmatrix}
\] \quad \text{P}_1 \text{ choices}

\[
P_2 \text{ choices}
\]

\[
B = \begin{bmatrix}
2 & 0 \\
30 & 8
\end{bmatrix}
\] \quad \text{P}_1 \text{ choices}

\[
P_2 \text{ choices}
\]

Solution to Exercise 10.1. The following MATLAB® code can be used to solve both problems.
Completely Mixed Nash Equilibria

Computation of Completely Mixed NE

Numerical Computation of MNE

Practice Exercises

```matlab
[m,n] = size(A);
x0 = rand(n+m+2,1);

% y'( A + b )z -p - q
H = [zeros(m,m), A+B, zeros(m,2); A'+B', zeros(n,n+2); zeros(2, m+n+2)];
c = [zeros(m+n,1); -1; -1];

% Az >= p & B'y >= q
Ain = [zeros(m,m), -A, ones(m,1), zeros(m,1); -B', zeros(n,n+1), ones(n,1)];
bin = zeros(m+n,1);

% sum(y) = sum(z)=1
Aeq = [ones(1,m), zeros(1,n+2); zeros(1,m), ones(1,n),0 ,0];
beq = [1;1];

% y.i , z.i in [0,1]
low = [zeros(n+m,1); -inf; -inf];
high = [ones(n+m,1);+inf; +inf];

% solve quadratic program
options = optimset('TolFun',1e-8,'TolX',1e-8,'TolPCG',1e-8,'Algorithm','active-set');
[x,val,exitflag ] = quadprog(H,c,Ain,bin,Aeq,beq,low,high,x0,options)

y = x(1:m)
z = x(m+1:m+n)
p = x(m+n+1)
q = x(m+n+2)
```

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Practice Exercises

1. For the BoS game, the code should be preceded by
   \[
   A = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix};
   \]
   and we obtain the following global minimum
   \[
   y = 0.3333 \\
   0.6667 \\
   z = 0.1667 \\
   0.8333 \\
   \]
   \[
   p = -0.3333 \\
   q = -0.3333 \\
   \]

2. For the prisoners’ dilemma, the code should be preceded by
   \[
   A = \begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix}; \quad B = \begin{bmatrix} 2 & 0 \\ 30 & 8 \end{bmatrix};
   \]
   and we obtain the following global minimum
   \[
   y = 0.0000 \\
   1.0000 \\
   z = 0.0000 \\
   1.0000 \\
   \]
   \[
   p = 8.0000 \\
   q = 8.0000 \\
   \]

15% of the time did we get a global minima.
The remaining 85% of the cases, we obtained local minima.
End of Lecture

10 - Computation of Nash Equilibria for Bimatrix Games

Questions?