Motivation
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Pure Policies and Saddle-Point Equilibria
Matrix Form for

COSC-6590/GSCS-6390
Games: Theory and Applications
Lecture 07 - Games in Extensive Form

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Motivation
Consider a zero-sum matrix game for which

\[
A = \begin{bmatrix}
1 & 3 & 0 \\
6 & 2 & 7
\end{bmatrix}
\]

\(P_1\) choices  
(Top, Bottom)

\(P_2\) choices  
(Left, Middle, Right)

For this game we have

\[
\bar{V}(A) = \max_i \min_j a_{ij} = 2
\]

\[
\underline{V}(A) = \min_i \max_j a_{ij} = 3
\]

\[
V_m(A) = \min_y \max_z y'Az = \max_z \min_y y'Az = \frac{8}{3} \approx 2.667
\]
Motivation

Each of these values (and policies) are meaningful for a particular information structure of the game:

- $V(A)$: outcome when the maximizer $P_2$ plays first (and the minimizer $P_1$ knows $P_2$’s choice before selecting an action).
- $\bar{V}(A)$: outcome when the minimizer $P_1$ plays first (and the maximizer $P_2$ knows $P_1$’s choice before selecting an action).
- $V_m(A)$: expected value of the outcome when both players play simultaneously (none knowing the others choice before selecting their actions).

However, the matrix description does not capture the information structure of the game

- in fact, other information structures are possible.
Extensive Form
An EFR of a zero-sum two-person game is a decision tree.

- Each **node** must be associated with one player ($P_1$ or $P_2$).
- The **links** emanating from one node correspond to decisions made by the player.
- All nodes must be enclosed in **dashed boxes**, called **information sets** (IS).
  Each IS may contain one or more nodes of the same player. All nodes in the same IS are indistinguishable for the corresponding player and must have the same alternatives.
- The different **leafs** of the tree are the final **outcomes** of the game and should be labeled with a cost/reward.
Extensive Form Representation

For $A$ there is a total of 6 possible information structures

$$1 + 1 + 1 + 3 = 6$$

$P_1$ plays first

$P_2$ plays first with full information

$P_2$ plays first with partial information

Alternative games in extensive form represented by $A$

(a) $P_1 - P_2$: $P_1$ plays first, and $P_2$ plays knowing $P_1$’s decision.

(b) Simultaneous: Both players play at the same time without knowing each others choice.

(c) $P_2 - P_1$: $P_2$ plays first, and $P_1$ plays knowing $P_2$’s decision.

(d) Mixed: $P_2$ plays first, if $P_2$ plays L then $P_1$ knows about this choice, but if $P_2$ plays M or R, then $P_1$ does not know which was played.

(e) Mixed: $P_2$ plays first, if $P_2$ plays R then $P_1$ knows about this choice, but if $P_2$ plays L or M, then $P_1$ does not know which was played.

(f) Mixed: $P_2$ plays first, if $P_2$ plays M then $P_1$ knows about this choice, but if $P_2$ plays L or R, then $P_1$ does not know which was played.
Multi-Stage Games
Multi-Stage Games

A sequence of rounds (or stages). In each stage the players have the opportunity to select one action.

EFR: tree can have more than two levels
  - generally twice as many levels as the number of stages.

(a) Two-stage game with different types of information sets, depending on the players’ actions.
Multi-Stage Games

Some multi-stage games have a **variable number of stages**, depending on the actions of the players.

Example: Chess

- some actions of the players will lead to the termination of the game with very few stages, whereas other sets of actions may require a large number of stages before the game is over.
**Multi-Stage Games**

**Information Sets** may span over different stages as long as they only contain nodes of the same player and all nodes within the set exhibit exactly the same possible actions for that player.

(c) Two stage game with an information set for P₂ that spans across stages: If P₁ initially chooses B, then P₂ must make a decision without knowing the stage of play (this would be a very “memory-constrained” player).

When an IS spans several stages, that particular player does not know at which stage play is taking place.
Pure Policies and Saddle-Point Equilibria
Pure Policies and Saddle-Point Equilibria

**Pure policy** for the player \( P_i \): a decision rule (i.e., a function) that associates one action to each IS of this player.

**Example:** for the \( P_1 - P_2 \) game

Possible policies \( \pi_1, \pi_2 \) for \( P_1 \) and \( P_2 \), respectively

\[
\pi_1(IS) = \begin{cases} 
T & \text{if } IS = \alpha \\
M & \text{if } IS = \beta \\
R & \text{if } IS = \xi
\end{cases}
\]

\[
\pi_2(IS) = \begin{cases} 
M & \text{if } IS = \beta \\
R & \text{if } IS = \xi
\end{cases}
\]

Total number of distinct pure policies for a given player is

\[
(# \text{ actions of 1st IS}) \times (# \text{ actions of 2nd IS}) \times \cdots \times (# \text{ actions of last IS})
\]

product over all information sets for that player
Pure Policies and Saddle-Point Equilibria

For the alternate play \( P_1 - P_2 \) game, we have a total of

- 2 pure policies for \( P_1 \), and
- \( 3 \times 3 = 9 \) pure policies for \( P_2 \).

For the simultaneous play game we have a total of

- 2 pure policies for \( P_1 \), and
- 3 pure policies for \( P_2 \).
Pure Policies and Saddle-Point Equilibria

Use the concepts of security levels, security policies, and saddle-point equilibria with the understanding that:

1. **Action spaces:** the sets $\Gamma_1$ and $\Gamma_2$ of all pure policies for players $P_1$ and $P_2$, respectively.

2. For a particular pair of pure policies $\gamma \in \Gamma_1$, $\sigma \in \Gamma_2$ we denote by $J(\gamma, \sigma)$ the **outcome of the game** when $P_1$ uses policy $\gamma$ and $P_2$ uses policy $\sigma$.

**Definition 7.1** (Pure saddle-point equilibrium). A pair of policies $(\gamma^*, \sigma^*) \in \Gamma_1 \times \Gamma_2$ is a pure saddle-point equilibrium if

\[
J(\gamma^*, \sigma^*) \leq J(\gamma, \sigma^*) \quad \forall \gamma \in \Gamma_1
\]

\[
J(\gamma^*, \sigma^*) \geq J(\gamma^*, \sigma) \quad \forall \sigma \in \Gamma_2
\]

and $J(\gamma^*, \sigma^*)$ is called the saddle-point value.
Matrix Form for Games in Extensive Form
A game in **extensive form** can be converted into an equivalent game in **matrix form** by regarding each policy of the game in extensive form as a possible action in a game in matrix form.

Game in extensive form, with sets as pure policies for $P_1$ and $P_2$:

$$
\Gamma_1 = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}, \quad \Gamma_2 = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}
$$

Define an $m \times n$ matrix $A_{\text{ext}}$, where

1. each row is one pure policy for $P_1$, and $m$ is the number of distinct pure policies in $\Gamma_1$,
2. each col is one pure policy for $P_2$, and $n$ is the number of distinct pure policies in $\Gamma_2$,
3. entry $a_{ij}$ is equal to the outcome of the game $J(\gamma_i, \sigma_j)$ when $P_1$ selects the pure policy $\gamma_i$ (the $i$th row) and $P_2$ selects the pure policy $\sigma_j$ (the $j$th column).
Matrix Form for Games in Extensive Form

Interpretation of matrix game $A_{\text{ext}}$: a game being played under simultaneous play
- players choose their policies before the game starts.

Recall: each row/column of $A_{\text{ext}}$ represents the choice of a policy, not an action.

Then, we can re-write the pure saddle-point equilibrium as

$$a_{ij^*} \leq a_{ij^*}, \quad \forall i$$
$$a_{i^*j} \geq a_{i^*j}, \quad \forall j$$

- $i^*$ is the row of $A_{\text{ext}}$ corresponding to the policy $\gamma^*$
- $j^*$ is the column of $A_{\text{ext}}$ corresponding to the policy $\sigma^*$.

Conclusion: $(\gamma^*, \sigma^*)$ is a pure saddle-point equilibrium if and only if $(i^*j^*)$ is a pure saddle-point equilibrium for $A_{\text{ext}}$. 
Matrix Form for Games in Extensive Form

**Proposition 7.1.**

The game in extensive form has a pure saddle-point equilibrium if and only if the matrix game defined by $A_{ext}$ has a pure saddle-point equilibrium

$$V(A_{ext}) = \bar{V}(A_{ext})$$

If such equilibria exist, they have the same saddle-point value.

**Note.** The size of the matrix $A_{ext}$ grows exponentially with the number of actions and information sets, which limits the use of this procedure to fairly small games.
Example 7.1. For the $P_1 - P_2$ game in extensive form suppose we enumerate the policies for $P_1$ and $P_2$ as

$$P_1 : \begin{array}{c|cc} IS & \gamma_1 & \gamma_2 \\ \hline \alpha & T & B \\ \end{array}$$

$$P_2 : \begin{array}{c|cccccccc} IS & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 \\ \hline \beta & L & L & L & M & M & M & R & R & R \\ \xi & L & M & R & L & M & R & L & M & R \\ \end{array}$$

(a) $P_1 - P_2$: $P_1$ plays first, and $P_2$ plays knowing $P_1$’s decision.
Matrix Form for Games in Extensive Form

In this case, we obtain

$$A_{\text{ext}} = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 3 & 0 & 0 & 0 \\ 6 & 2 & 7 & 6 & 2 & 7 & 6 & 2 & 7 \end{bmatrix}$$

\{P_1 \text{ policies} \ (\gamma_1, \gamma_2) \}

\{P_2 \text{ policies} \ (\sigma_1, \sigma_2, \ldots, \sigma_9) \}

For this matrix:

- $V(A_{\text{ext}}) = \bar{V}(A_{\text{ext}}) = 3$.
- Two pure saddle-point equilibria: (1, 4) and (1, 6).

Consequently, the $P_1 - P_2$ game in extensive form also has two pure saddle-point equilibria: $(\gamma_1, \sigma_4)$ and $(\gamma_1, \sigma_6)$.

- But $(\gamma_1, \sigma_6)$ is the dominating saddle-point equilibrium.
Matrix Form for Games in Extensive Form

Indeed

\[ A_{\text{ext}} = \begin{bmatrix} \text{col. 9 dominates over 7 & 8} \\ \text{(col. 3 dominates over 1 & 2, col. 6 dominates over 4 & 5)} \end{bmatrix} \rightarrow A^\dagger_{\text{ext}} := \begin{bmatrix} 1 & 3 & 0 \\ 7 & 7 & 7 \end{bmatrix} \]

\[ \text{row 1 dominates over 2} \rightarrow A^\ddagger_{\text{ext}} := [1 \ 3 \ 0] \quad \text{col. 2 dominates over 1 & 3} \rightarrow A^\flat_{\text{ext}} := [3] \]

This corresponds to the pure saddle-point equilibrium \((\gamma_1, \sigma_6)\)

\[ \gamma_1(IS) = \begin{cases} T & \text{if IS} = \alpha \\ M & \text{if IS} = \beta \quad \text{(i.e., if } P_1 \text{ chooses T)} \end{cases} \]

\[ \sigma_6(IS) = \begin{cases} \begin{align*} M & \text{if IS} = \beta \quad \text{(i.e., if } P_1 \text{ chooses T)} \\ L & \text{if IS} = \xi \quad \text{(i.e., if } P_1 \text{ chooses B)} \end{align*} \]

However, this does not invalidate the fact that

\[ \gamma_1(IS) = \begin{cases} T & \text{if IS} = \alpha \\ M & \text{if IS} = \beta \quad \text{(i.e., if } P_1 \text{ chooses T)} \end{cases} \]

\[ \sigma_4(IS) = \begin{cases} \begin{align*} M & \text{if IS} = \beta \quad \text{(i.e., if } P_1 \text{ chooses T)} \\ L & \text{if IS} = \xi \quad \text{(i.e., if } P_1 \text{ chooses B)} \end{align*} \]

is also a (non-dominating) saddle-point equilibrium.
Recursive Computation of Equilibria for Single-Stage Games
Recursive Computation of Eq. for Single-Stage Games

Procedure starts at the bottom of the tree and moves upwards
- this one is computationally much more attractive.

Procedure for single-stage games (assume $P_1$ is first-acting)

**Step 1.** Construct a matrix game corresponding to each information set of $P_2$. Each matrix game will have
- one action of $P_1$ for each edge entering the information set
- one action of $P_2$ for each edge leaving the information set.

**Step 2.** Compute pure saddle-point equilibria for each of the matrix games constructed.
- method fails if none has a pure saddle-point equilibrium.
Otherwise, $P_2$’s pure security/saddle-point policy should map to each information set the action corresponding to $P_2$’s pure security policy in the corresponding matrix game.
Step 3. Replace each information set by the value of the corresponding pure saddle-point equilibrium.

- link leading to this value should be labeled with $P_1$’s pure policy in the corresponding matrix game.

Step 4. Player $P_1$ chooses the policy corresponding to the pure saddle-point value that is most favorable for $P_1$.

This procedure is guaranteed to result in a pure saddle-point equilibrium.
Examples illustrating how to apply this procedure

1.- Initial Game

\[ A_\beta = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \]  

\( P_1 \) choices  
\( (L, M) \)  
\( P_2 \) choices  
\( (T, B) \)  

2. Construction of matrix games for the different information sets

\[ A_\xi = \begin{bmatrix} -1 & 0 \end{bmatrix} \]  

\( P_1 \) choices  
\( (R) \)  

\( P_2 \) choices  
\( (T, B) \)
Examples illustrating how to apply this procedure

3. Solution of matrix games for the different information sets.

\[ V(A_\beta) = 1, \quad \bar{V}(A_\beta) = 2 \]

\[ V(A_\xi) = \bar{V}(A_\xi) = 0 \]

The method fails

- there is no pure saddle-point equilibrium for \( A_\beta \).
Examples illustrating how to apply this procedure

1.- Initial Game

2. Construction of matrix games for the different information sets

\[ A_\beta = \begin{bmatrix} 3 & 1 \end{bmatrix} \quad \text{P}_1 \text{ choices} \quad \left( L \right) \]

\[ A_\xi = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad \text{P}_1 \text{ choices} \quad \left( M, R \right) \]
Examples illustrating how to apply this procedure

3. Solution of matrix games for the different information sets.

\[ \overline{V}(A_\beta) = \bar{V}(A_\beta) = 3, \quad (i^*, j^*) = (L, T) \]
\[ \overline{V}(A_\xi) = \bar{V}(A_\xi) = 0, \quad (i^*, j^*) = (R, B) \]

\[ P_2 \text{ policy} = \begin{cases} 
T & \text{if IS} = \beta \\
B & \text{if IS} = \xi 
\end{cases} \]

4. Replacement of information sets by the values of the games.

\[ P_1 \text{ policy} = \{ R \quad \text{if IS} = \alpha \]
Proposition 7.2. For any zero-sum single-stage game in extensive form, if every matrix game corresponding to the information sets of the second-acting player has a pure saddle-point equilibrium, then the game in extensive form has a pure saddle-point equilibrium.

When all the information sets of a game have a single element, the matrix games corresponding to these sets necessarily have a pure saddle-point equilibrium because one of the players only has a single action.

Corollary 7.1. Every zero-sum single-stage game in extensive form for which all information sets have a single element has a pure saddle-point equilibrium.
Attention! Policies constructed with this procedure are guaranteed to be saddle-point equilibria:

- procedure may not bring up all possible pure saddle-point equilibria
- it will only produce dominating saddle-point equilibria.

When the procedure fails, there may or there may not be pure saddle-point equilibria.
Recursive Computation of Eq. for Single-Stage Games

1.- Initial Game

2. Construction of matrix games for the different information sets

\[ A_\beta = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix} \] \quad \begin{cases} P_1 \text{ choices} \\ (L, M) \end{cases} \quad A_\xi = \begin{bmatrix} 0 & 7 \end{bmatrix} \] \quad \begin{cases} P_1 \text{ choices} \\ (R) \end{cases}
Recursive Computation of Eq. for Single-Stage Games

3.- Solution of matrix games for the different information sets.

\[ V(A_\beta) = 2, \quad \bar{V}(A_\beta) = 3, \]
\[ V(A_\xi) = \bar{V}(A_\xi) = 7 \]

The method fails

- there is no pure saddle-point equilibrium for \( A_\beta \).

This game has no pure saddle-point equilibria.
Recursive Computation of Eq. for Single-Stage Games

1.- Initial Game

2. Construction of matrix games for the different information sets

\[ A_\beta = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix} \]  
\( P_1 \) choices \((L, M)\)  
\( P_2 \) choices \((T, B)\) 

\[ A_\xi = \begin{bmatrix} -1 & 0 \end{bmatrix} \]  
\( P_1 \) choices \((R)\)  
\( P_2 \) choices \((T, B)\)
Recursive Computation of Eq. for Single-Stage Games

3.- Solution of matrix games for the different information sets.

\[
\begin{align*}
V(A_\beta) &= 2, & \bar{V}(A_\beta) &= 3, \\
\underline{V}(A_\xi) &= \bar{V}(A_\xi) = 0
\end{align*}
\]

The method fails

- there is no pure saddle-point equilibrium for \( A_\beta \).

However, this game has several pure saddle-point equilibria, including

\[
P_2 \text{ policy } = \begin{cases} 
T & \text{if } IS = \beta \\
B & \text{if } IS = \xi 
\end{cases} \quad P_1 \text{ policy } = \begin{cases} 
R & \text{if } IS = \alpha 
\end{cases}
\]

\( P_1 \) never choose \( L \) or \( M \) since \( R \) strictly dominates both choices.
Feedback Games
Feedback Games

Definition 7.2 (Feedback games).
A multi-stage game in extensive form is a feedback game (in extensive form) if the following conditions hold:

C1: no information set spans over multiple stages,
   • when a player must select an action, they know the current stage of the game.

C2: the nodes that correspond to the start of each stage are the roots of sub-trees that do not share information sets with each other (including at the level of the root).
   • both players have full information about what both players did in past stages of the game.
Feedback Games - Examples

(a) Feedback game.

(b) Not a feedback game because the dash-dot information set spans over multiple stages, violating condition C7.1.

(c) Not a feedback game because at the start of the second stage $P_1$ does not necessarily know the action selected by $P_2$ in the first stage. The dash-dot information set violates condition C7.2.

(d) Not a feedback game because at the second stage, $P_2$ does not necessarily know the action selected by $P_1$ at the first stage. The dash-dot information set violates condition C7.2.
Feedback Saddle-Point for Multi-Stage Games
Feedback Saddle-Point for Multi-Stage Games

In feedback games IS do not span over multiple stages
- then, we can decompose a policy for a particular player into several sub-policies, one for each stage.

Consider a feedback game with $K$ stages, and denote by $\mathcal{I}$ the set of all information sets for player $P_i$.

$\mathcal{I}$ can be partitioned into $K$ disjoint subsets

$$\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_K, \quad \mathcal{I} = \bigcup_{i=1}^{K} \mathcal{I}_i$$

each one containing all the Information Sets for a specific stage.
Feedback Saddle-Point for Multi-Stage Games

If we are then given a policy

$$\gamma : \mathcal{I} \rightarrow \mathcal{A}, \quad \alpha \mapsto \gamma(\alpha)$$

that maps each IS in \( \mathcal{I} \) to some action in the action space \( \mathcal{A} \), we can decompose \( \gamma \) into \( K \) sub-policies

$$\gamma := \{\gamma_1, \gamma_2, \ldots, \gamma_K\},$$

where each

$$\gamma_i : \mathcal{I}_i \rightarrow \mathcal{A}, \quad \alpha \mapsto \gamma(\alpha)$$

only maps the \( i \)th stage information sets to actions in \( \mathcal{A} \).

This decomposition motivates a new definition of saddle-point equilibria for feedback games.
**Feedback Saddle-Point for Multi-Stage Games**

**Definition 7.3** (Feedback pure saddle-point equilibrium).

For a feedback game with $K$ stages, a pair of policies $(\gamma^*, \sigma^*)$

$$
\gamma^* := \{\gamma_1^*, \gamma_2^*, \ldots, \gamma_K^*\}, \quad \sigma^* := \{\sigma_1^*, \sigma_2^*, \ldots, \sigma_K^*\}
$$

is a **feedback pure saddle-point equilibrium** if for every stage $k$ and every policy $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ and $\sigma_1, \sigma_2, \ldots, \sigma_{k-1}$ for the stages prior to $k$ we have that

$$
J \left( \left\{ \gamma_{1\ldots k-1}, \begin{smallmatrix} \gamma_k \end{smallmatrix}, \gamma_{k+1\ldots K} \right\}, \left\{ \sigma_{1\ldots k-1}, \begin{smallmatrix} \sigma_k \end{smallmatrix}, \sigma_{k+1\ldots K} \right\} \right)
\leq J \left( \left\{ \gamma_{1\ldots k-1}, \begin{smallmatrix} \gamma_k \end{smallmatrix}, \gamma_{k+1\ldots K} \right\}, \left\{ \sigma_{1\ldots k-1}, \begin{smallmatrix} \sigma_k \end{smallmatrix}, \sigma_{k+1\ldots K} \right\} \right) \quad \forall \gamma_k
$$

$$
J \left( \left\{ \gamma_{1\ldots k-1}, \begin{smallmatrix} \gamma_k \end{smallmatrix}, \gamma_{k+1\ldots K} \right\}, \left\{ \sigma_{1\ldots k-1}, \begin{smallmatrix} \sigma_k \end{smallmatrix}, \sigma_{k+1\ldots K} \right\} \right)
\geq J \left( \left\{ \gamma_{1\ldots k-1}, \begin{smallmatrix} \gamma_k \end{smallmatrix}, \gamma_{k+1\ldots K} \right\}, \left\{ \sigma_{1\ldots k-1}, \begin{smallmatrix} \sigma_k \end{smallmatrix}, \sigma_{k+1\ldots K} \right\} \right) \quad \forall \sigma_k
$$

Universal quantifications refer to all $k$ stage pure sub-policies.
Feedback Saddle-Point for Multi-Stage Games

\[
J \left( \{ \gamma_{1\ldots k-1}, \gamma_k^*, \gamma_{k+1\ldots K} \}, \{ \sigma_{1\ldots k-1}, \sigma^*_k, \sigma_{k+1\ldots K} \} \right) \\
\leq J \left( \{ \gamma_{1\ldots k-1}, \gamma_k, \gamma_{k+1\ldots K} \}, \{ \sigma_{1\ldots k-1}, \sigma^*_k, \sigma_{k+1\ldots K} \} \right) \quad \forall \gamma_k \\
J \left( \{ \gamma_{1\ldots k-1}, \gamma_k^*, \gamma_{k+1\ldots K} \}, \{ \sigma_{1\ldots k-1}, \sigma^*_k, \sigma_{k+1\ldots K} \} \right) \\
\geq J \left( \{ \gamma_{1\ldots k-1}, \gamma_k^*, \gamma_{k+1\ldots K} \}, \{ \sigma_{1\ldots k-1}, \sigma_k, \sigma^*_{k+1\ldots K} \} \right) \quad \forall \sigma_k
\]

In words:

No matter what the players do before stage \( k \) (rational or not) and assuming that both players will play at the equilibrium after stage \( k \), the stage \( k \) sub-policies \((\gamma^*_k, \sigma^*_k)\) must be a pure saddle-point equilibrium.
Lemma 7.1. For every feedback game in extensive form, a feedback pure saddle-point equilibrium in the sense of Definition 7.3 is always a pure saddle-point equilibrium in the sense of Definition 7.1.

Proof of Lemma 7.1. To prove this result, show that

\[ J \left( \{ \gamma_{1\ldots k-1}, \overline{\gamma_k}, \gamma_{k+1\ldots K} \}, \{ \sigma_{1\ldots k-1}, \overline{\sigma_k}, \sigma_{k+1\ldots K} \} \right) \]

\[ \leq J \left( \{ \gamma_{1\ldots k-1}, \overline{\gamma_k}, \gamma_{k+1\ldots K} \}, \{ \sigma_{1\ldots k-1}, \overline{\sigma_k}, \sigma_{k+1\ldots K} \} \right) \forall \gamma_k \]

\[ J \left( \{ \gamma_{1\ldots k-1}, \overline{\gamma_k}, \gamma_{k+1\ldots K} \}, \{ \sigma_{1\ldots k-1}, \overline{\sigma_k}, \sigma_{k+1\ldots K} \} \right) \]

\[ \geq J \left( \{ \gamma_{1\ldots k-1}, \overline{\gamma_k}, \gamma_{k+1\ldots K} \}, \{ \sigma_{1\ldots k-1}, \overline{\sigma_k}, \sigma_{k+1\ldots K} \} \right) \forall \sigma_k \]

actually imply that

\[ J(\gamma^*, \sigma^*) \leq J(\sigma, \gamma^*), \forall \gamma \]

\[ J(\gamma^*, \sigma^*) \geq J(\gamma^*, \sigma), \forall \sigma \]
Feedback Saddle-Point for Multi-Stage Games

Previous equations can be re-written in terms of sub-policies as:

\[
J\left(\{\gamma_1^*, \gamma_2^*, \ldots, \gamma_K^*\}, \{\sigma_1^*, \sigma_2^*, \ldots, \sigma_K^*\}\right) \leq J\left(\{\gamma_1, \gamma_2, \ldots, \gamma_K\}, \{\sigma_1, \sigma_2, \ldots, \sigma_K\}\right), \quad \forall \gamma_1, \gamma_2, \ldots, \gamma_K
\]

\[
J\left(\{\gamma_1^*, \gamma_2^*, \ldots, \gamma_K^*\}, \{\sigma_1^*, \sigma_2^*, \ldots, \sigma_K^*\}\right) \leq J\left(\{\gamma_1^*, \gamma_2^*, \sigma_3^*, \ldots, \gamma_K^*\}, \{\sigma_1^*, \sigma_2^*, \sigma_3^*, \ldots, \sigma_K^*\}\right), \quad \forall \sigma_1, \sigma_2, \ldots, \sigma_K
\]

To accomplish this, use the first eq., for the first stage \(k = 1\)

\[
J\left(\{\gamma_1^*, \gamma_2^*\}, \{\sigma_1^*, \sigma_2^*\}\right) \leq J\left(\{\gamma_1, \gamma_2, \sigma_3, \ldots, \gamma_K\}, \{\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_K\}\right), \quad \forall \gamma_1
\]

and now for the stage \(k = 2\), with \(\sigma_1 := \sigma_1^*\) in the first stage

\[
J\left(\{\gamma_1, \gamma_2^*, \gamma_3^*\}, \{\sigma_1^*, \sigma_2^*, \sigma_3^*\}\right) \leq J\left(\{\gamma_1, \gamma_2, \gamma_3^*\}, \{\sigma_1^*, \sigma_2^*, \sigma_3^*\}\right), \quad \forall \gamma_2
\]
Feedback Saddle-Point for Multi-Stage Games

Combining these two inequalities we obtain

\[ J(\{\gamma_1^*...K\}, \{\sigma_1^*...K\}) \leq J(\{\gamma_1, \gamma_2, \gamma_3^*...K\}, \{\sigma_1^*...K\} x), \quad \forall \gamma_1, \gamma_2 \]

As we combine this with the first eq. in Lemma 7.1, for each subsequent stage \( k \in \{3, 4, \ldots, K\} \) and with \( \sigma_i^* = \sigma_i^* \), \( \forall i \in \{1, 2, \ldots, k - 1\} \) for the previous stages, we eventually obtain

\[ J(\{\gamma_1^*...K\}, \{\sigma_1^*...K\}) \leq J(\{\gamma_1...K\}, \{\sigma_1^*...K\} x), \quad \forall \gamma_1, \gamma_2, \ldots, \gamma_K \]

which proves that \( J(\gamma^*, \sigma^*) \leq J(\gamma, \sigma^*), \quad \forall \gamma \).

The other saddle-point inequality presented, can be proved similarly using the second eq. in Lemma 7.1.
Feedback Saddle-Point for Multi-Stage Games

Note 8 (Feedback pure saddle-point equilibria).

Why introduce a new notion of saddle-point equilibria when we already had a very reasonable one?

There are several reasons:

1. Finding feedback saddle-point equilibria is easier than finding saddle-point equilibria that are not feedback.

2. Feedback saddle-point equilibria are better:
   - they provide optimal security levels, even if the player did not play rationally in the past.
   - these security levels may not be as good as the ones that could have been obtained if the player had been playing rationally since the beginning of the game.
Recursive Computation of Equilibria for Multi-Stage Games
Feedback Saddle-Point for Multi-Stage Games

Procedure: (assume $P_1$ is first-acting, game has $K$ stages)

Step 1. Construct a matrix game corresponding to each IS of $P_2$ at the $K$th stage. Each matrix game will have one action of $P_1$ for each edge entering the IS and one action of $P_2$ for each edge leaving the IS.

Step 2. Compute pure saddle-point equilibria for each of the matrix games constructed.

- If any of these matrix games does not have a pure saddle-point equilibrium, this method fails.
- Otherwise, $P_2$’s pure security/saddle-point sub-policy $\sigma^*_K$ for the $K$th stage should map to each IS the action corresponding to $P_2$’s security policy/saddle-point in the corresponding matrix game.
Feedback Saddle-Point for Multi-Stage Games

**Step 3.** Replace each IS for $P_2$ at the $K$th stage by the value of the corresponding pure saddle-point equilibrium.

- Link leading to this value should be labeled with $P_1$’s pure security/saddle-point in the corresponding matrix game.

**Step 4.** The pure security/saddle-point sub-policy $\gamma^*_K$ for $P_1$ at $K$th stage maps to each IS of $P_1$, the action in the link corresp. to the most favorable pure saddle-point value for $P_1$.

The policies $(\gamma^*_K, \sigma^*_K)$ have the property that, for every policy $\gamma_1, \gamma_2, \ldots, \gamma_{K-1}$ and $\sigma_1, \sigma_2, \ldots, \sigma_{K-1}$ for the stages prior to $K$,

$$J \left( \left( \{\gamma_1 \ldots K-1, \gamma^*_K \}, \{\sigma_1 \ldots K-1, \sigma^*_K \} \right) \right) \leq J \left( \left( \{\gamma_1 \ldots K-1, \gamma_K \}, \{\sigma_1 \ldots K-1, \sigma_K \} \right) \right),$$

$$J \left( \left( \{\gamma_1 \ldots K-1, \gamma^*_K \}, \{\sigma_1 \ldots K-1, \sigma^*_K \} \right) \right) \geq J \left( \left( \{\gamma_1 \ldots K-1, \gamma^*_K \}, \{\sigma_1 \ldots K-1, \sigma_K \} \right) \right),$$

as required for a feedback pure saddle-point equilibrium.
Step 5. Replace each IS for $P_1$ at the $K$th stage by the value corresponding to the link selected by $P_1$.

At this point we have a game with $K - 1$ stages, and we compute the sub-policies $(\gamma^*_{K-1}, \sigma^*_{K-1})$ using the exact same procedure described above.

This algorithm is iterated until all sub-policies have been obtained.

At every stage we guarantee by construction that the conditions in the equations, for a feedback pure saddle-point equilibrium hold.
Feedback Saddle-Point for Multi-Stage Games

Illustration of steps 3-5 in the recursive computation of equilibria for multi-stage games.

\( \alpha \): information state corresponding to \( P_2 \) at the \( K \)th stage  
\( V(\alpha) \): value of the matrix game associated with that information state  
\((i^*, j^*)\): the corresponding saddle-point equilibrium point.
Feedback Saddle-Point for Multi-Stage Games

**Proposition 7.3.** For any zero-sum game in extensive form with a finite number of stages, if every matrix game constructed using the above procedure has a pure saddle-point equilibrium, then the overall game in extensive form has a feedback pure saddle-point equilibrium.

When all the IS of a game have a single element, the matrix games corresponding to these sets necessarily have a saddle-point equilibrium because one of the players only has a single action.

This observation leads to the following consequence:

**Corollary 7.2.** Every zero-sum game in extensive form with a finite number of stages for which all information sets have a single element has a feedback pure saddle-point equilibrium.
Practice Exercise
Practice Exercise

7.1 (Number of games in extensive form).

How many different extensive games are possible for a given $m \times n$ matrix of outcomes?

Solution to Exercise 7.1.

Assuming $P_1$ plays first, the number of distinct games is given by the **Bell number** $b_m$: the number of partitions of the set \{1, 2, \ldots, m\}. $b_m$ is computed recursively by

$$b_{k+1} = \sum_{i=1}^{k} \binom{k}{i}$$

(cf. Table below).

<table>
<thead>
<tr>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>$\cdots$</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_m$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>15</td>
<td>52</td>
<td>203</td>
<td>877</td>
<td>4149</td>
<td>21147</td>
<td>$\cdots$</td>
<td>$\approx 1.4 \times 10^9$</td>
</tr>
</tbody>
</table>
Practice Exercise

On the other hand, assuming that player $P_2$ plays first, the number of distinct games is given by the Bell number $b_n$. Consequently, the total number of distinct games is equal to

$$b_m - b_n - 1$$

subtraction by 1 is to avoid double counting the simultaneous play, which otherwise would be counted twice. For example for the $2 \times 3$ matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 6 & 2 & 7 \end{bmatrix}$$

$P_1$ choices

$P_2$ choices

Top, Bottom

Left, Middle, Right

we have $b_2 + b_3 - 1 = 2 + 5 - 1 = 6$ games.
End of Lecture

07 - Games in Extensive Form

Questions?