COSC-6590/GSCS-6390

Games: Theory and Applications

Lecture 06 - Computation of Mixed Saddle-Point Equilibrium Policies

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Graphical Method
Graphical Method

To find mixed saddle-point equilibria
- compute mixed security policies for both players

For $2 \times 2$ games we can use the graphical method

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}$$

Compute the mixed security policy for $P_1$

$$\min_y \max_z y'Az = \min_y \max_z y_1 y_2 \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \min_y \max_z z_1(3y_1 - y_2) + z_2(y_2)$$

$$= \min_y \max \{3y_1 - y_2, y_2\}$$
Graphical Method

Since \( y_1 + y_2 = 1 \), we must have \( y_2 = 1 - y_1 \) and therefore

\[
\min \max y' Az = \min \max \{4y_1 - 1, 1 - y_1\}. 
\]

To find the optimal value for \( y_1 \)

- draw the two lines \( 4y_1 - 1 \) and \( 1 - y_1 \) in the same axis
- pick the maximum point-wise
- select the point \( y_1^* \) for which the maximum is smallest.

Point is the security policy. Maximum is the value of the game.

\[
\min \max y' Az = \frac{3}{5},
\]

\[
y_1^* = \frac{2}{5} \Rightarrow y^* = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix}'.
\]
Graphical Method

Compute the mixed security policy for $P_2$

$$\max_{z=[z_1 \ z_2]} \ \min_{y=[y_1 \ y_2]} \ y' Az = \max_{z} \ \min_{y} [y_1 \ y_2] \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \max_{z} \ \min_{y} y_1 (3z_1) + y_2 (-z_1 + z_2)$$

$$= \max_{z} \ \min \{3z_1, -z_1 + z_2\}$$

This results in

$$\max_{z} \ \min_{y} y' Az = \frac{3}{5},$$

$$z_1^* = \frac{1}{5} \Rightarrow z^* = \begin{bmatrix} 1 \\ \frac{4}{5} \end{bmatrix}'$$
Linear Program Solution
Linear Program Solution

Systematic numerical procedure to find mixed saddle-point equilibria.

Goal is to compute

$$V_m(A) = \min_{y \in \mathcal{Y}} \max_{z \in \mathcal{Z}} y' Az = \max_{z \in \mathcal{Z}} \min_{y \in \mathcal{Y}} y' Az$$

where

$$\mathcal{Y} := \left\{ y \in \mathbb{R}^m : \sum_i y_i = 1, \ y_i \geq 0, \ \forall i \right\}$$

$$\mathcal{Z} := \left\{ z \in \mathbb{R}^n : \sum_j z_j = 1, \ z_j \geq 0, \ \forall j \right\}$$
Linear Program Solution

Compute the \textbf{inner max} in the $\min_y \max_z$ optimization.

For a fixed $y \in \mathcal{Y}$ we have

$$\max_{z \in \mathbb{Z}} y'Az = \max_{z \in \mathbb{Z}} \sum_{ij} y_i a_{ij} z_j = \max_{z \in \mathbb{Z}} \sum_j \left( z_j \sum_i y_i a_{ij} \right) = \max_j \sum_i y_i a_{ij}$$

Use an equality to convert a maximization into a constrained minimization: given a set of numbers $x_1, x_2, \ldots, x_m$,

$$\max_j x_j = \min \{ v \in \mathbb{R} : v \geq x_j, \ \forall j \}$$

Using in the previous equation we conclude that

$$\max_{z \in \mathbb{Z}} y'Az = \min \left\{ v : v \geq \sum_i y_i a_{ij} , \forall j \right\}$$

\textit{j}th entry of row vector $y'A$, or
\textit{j}th entry of column vector $A'y$
Linear Program Solution

Denoting by $\mathbf{1}$ a column vector consisting of ones, we re-write the condition in the set above as

$$v\mathbf{1} \geq \begin{bmatrix} \sum_i y_i a_{i1} \\ \sum_i y_i a_{i2} \\ \vdots \\ \sum_i y_i a_{in} \end{bmatrix} = A'y$$

This allows us to re-write $V_m(A)$ as a linear program

$$V_m(A) = \min_{y \in \mathcal{Y}} \min \{ v : v\mathbf{1} \geq A'y \}$$

$$= \begin{cases} \text{minimum} & v \\ \text{subject to} & y \geq 0 \\ & 1'y = 1 \\ & A'y \leq v\mathbf{1} \end{cases}$$

optimization over $m+1$ parameters $(v, y_1, y_2, \ldots, y_m)$
MATLAB® Hint 2.

Linear programs can be solved numerically with matlab using \texttt{linprog} from the Optimization toolbox or the freeware Disciplined Convex Programming toolbox, also known as CVX.

Solving this optimization, we obtain the value of the game $v^*$ and a mixed security policy $y^*$ for $P_1$.

Since the security policies are those that achieve the minimum in $V_m(A)$, once we have the value of the game we can obtain the set of all mixed security policies using

\[
\{ y \in \mathbb{R}^m : y \geq 0, \quad 1'y = 1, \quad v^*1 \geq A'y \}\]
Focusing on the $\max_z \min_y$ optimization, we conclude that

$$\min_y y^\prime Az = \cdots = \max \{ v : v1 \leq Az \}$$

Therefore

$$V_m(A) = \begin{array}{c}
\text{maximum} \\ v
\end{array} \text{ subject to } \begin{cases} \\
 z \geq 0 \\ \mathbf{1}z = 1 \\ A\mathbf{z} \geq v\mathbf{1}
\end{cases} \quad z \in \mathcal{Z}$$

optimization over $n+1$ parameters $(v, z_1, z_2, \ldots, z_n)$

Solving this optimization, we obtain the value of the game $v^*$ and a mixed security policy $z^*$ for $P_2$.

And we can obtain the set of all mixed security policies using

$$\{ z \in \mathbb{R}^n : z \geq 0, \quad \mathbf{1}^\prime z = 1, \quad v^* \mathbf{1} \leq Az \}$$
Linear Programs with MATLAB
MATLAB® Hint 2. (Linear programs).

\[ [x, \text{val}] = \text{linprog} (c, Ain, bin, Aeq, beq, low, high) \]

from MATLAB®’s Optimization Toolbox numerically solves linear programs of the form

\[
\begin{align*}
\text{minimum} & \quad c'x \\
\text{subject to} & \quad \text{Ain } x \leq \text{ bin} \\
& \quad \text{Aeq } x = \text{ beq} \\
& \quad \text{low } \leq x \leq \text{ high}
\end{align*}
\]

\text{val}: value of the minimum.
\text{x}: vector that achieves the minimum

To avoid the corresponding inequality constraints
- the vector \text{low} can have some or all entries equal to \(-\text{Inf}\)
- the vector \text{high} can have some or all entries equal to \text{Inf}
Linear Programs with MATLAB

Same optimization performed with the Disciplined Convex Programming (CVX) Toolbox

```matlab
cvx_begin
    variables x(size(Ain,2))
    minimize c*x
    subject to
        Ain*x <= bin;
        Aeq*x = beq;
        x >= low;
        x <= high;

cvx_end
```

CVX syntax is especially intuitive.
CVX code to find the value of a game defined by a matrix $A$
- and the mixed value and security policy for $P_1$ (left)
- and the mixed value and security policy for $P_2$ (right)

```matlab
CVX code to find the value of a game defined by a matrix $A$
- and the mixed value and security policy for $P_1$ (left)
- and the mixed value and security policy for $P_2$ (right)

```
Strictly Dominating Policies
Strictly Dominating Policies

Consider a game specified by an $m \times n$ matrix $A$.

- $m$ actions for $P_1$, and $n$ actions for $P_2$.

\[
A = \begin{bmatrix}
  \vdots & \cdots & a_{ij} & \cdots \\
  \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix} \tag{P_1 \text{ choices (minimizer)}}
\]

\[
P_2 \text{ choices (maximizer)}
\]
Strictly Dominating Policies

We say that row $i$ strictly dominates row $k$ if

$$a_{ij} < a_{kj} \quad \forall j$$

which means that no matter what $P_2$ does, the minimizer $P_1$ is always better off by selecting row $i$ instead of row $k$.

In practice, this means that

- Pure policies: $P_1$ will never select row $k$
- Mixed policies: $P_1$ will always select row $k$ with probability zero, i.e., $y_k^* = 0$ for any security/saddle-point policy.
Strictly Dominating Policies

Conversely, we say that column $j$ strictly dominates column $l$ if

$$a_{ij} > a_{il} \quad \forall i$$

which means that no matter what $P_1$ does, the maximizer $P_2$ is always better off by selecting column $j$ instead of column $l$.

In practice, this means that

- Pure policies: $P_2$ will never select column $l$
- Mixed policies: $P_1$ will always select col $l$ with probability zero, i.e., $z_i^* = 0$ for any security/saddle-point policy.
Strictly Dominating Policies

Finding dominating rows/columns in $A$ allows one to reduce the size of the problem that needs to be solved, as we can:

1. remove any rows/columns that are strictly dominated
2. compute (pure or mixed) saddle-point equilibria for the smaller game
3. recover the saddle-point equilibria for the original:
   - pure policies: saddle-point equilibria are the same, modulo some re-indexing to account for the fact that indexes of the rows/columns may have changed
   - mixed policies: may need to insert zero entries corresponding to the columns/rows that were removed.

By removing strictly dominated rows/columns one cannot lose security policies so all security policies for the original (larger) game correspond to security policies for the reduced game.
Strictly Dominating Policies

**Example 6.1** (Strictly dominating policies).

\[
A = \begin{bmatrix}
3 & -1 & 0 & -1 \\
4 & 1 & 2 & 0 \\
-1 & 0 & 1 & 0
\end{bmatrix}
\]

\[
\begin{align*}
P_1 \text{ choices (minimizer)} & : P_2 \text{ choices (maximizer)} \\
\end{align*}
\]

Since the 2nd row is strictly dominated by the 1st row

\[
A^\dagger = \begin{bmatrix}
3 & -1 & 0 & -1 \\
-1 & 0 & 1 & 0
\end{bmatrix}
\]

We now observe that both the 2nd and 4th column are (strictly) dominated by the 3rd column

\[
A^\ddagger = \begin{bmatrix}
3 & 0 \\
-1 & 1
\end{bmatrix}
\]
Strictly Dominating Policies

We found the following value and mixed security/saddle-point equilibrium policies

\[ V(A) = \frac{3}{5}, \quad y^* = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \quad z^* = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix}, \]

We thus conclude that the original game has the following value and mixed security/saddle-point equilibrium policies

\[ V(A) = \frac{3}{5}, \quad y^* = \begin{bmatrix} \frac{2}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}, \quad z^* = \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix}, \]
“Weakly” Dominating Policies
“Weakly” Dominating Policies

We say that row $i$ (weakly) dominates row $k$ if

$$a_{ij} \leq a_{ik} \quad \forall j$$

which means that no matter what $P_2$ does, the minimizer $P_1$ loses nothing by selecting row $i$ instead of row $k$.

We say that column $j$ (weakly) dominates column $l$ if

$$a_{ij} \geq a_{il} \quad \forall l$$

which means that no matter what $P_1$ does, the maximizer $P_2$ loses nothing by selecting column $j$ instead of column $l$. 
“Weakly” Dominating Policies

Remove weakly dominated rows/columns and be sure that

- the value of the reduced game is the same as the value of the original game, and
- one can reconstruct security policies/saddle-point equilibria for the original game from those for the reduced game.

One may lose some security policies/saddle-point equilibria that were available for the original game but that have no direct correspondence in the reduced game.

A game is said to be **maximally reduced** when no row or column dominates another one.

Saddle-point equilibria of maximally reduced games are called **dominant**.
“Weakly” Dominating Policies

**Example 6.2** (Weakly dominating policies).

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \left\{ \begin{array}{c} P_1 \text{ choices} \\ P_2 \text{ choices} \end{array} \right\} \]

Game has value \( V(A) = 1 \) and two pure saddle-point equilibria \((1, 2)\) and \((2, 2)\). However, this game can be reduced as follows:

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \xrightarrow{\text{2nd row dominates 1st}} A^\dagger := \begin{bmatrix} -1 & 1 \end{bmatrix} \xrightarrow{\text{2nd col dominates 1st}} A^\ddagger := [1] \]

from which one obtains the pure saddle-point equilibrium \((2, 2)\).

- i.e., the \( \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \) pair of mixed policies.
“Weakly” Dominating Policies

Alternatively, this game can also be reduced as follows:

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{2nd col dominates 1st} \quad \rightarrow \quad A^\dagger := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{1st row dominates 2nd} \quad \rightarrow \quad A^\ddagger := [1]
\]

from which one obtains the pure saddle-point equilibrium \((1, 2)\).
- i.e., the \(\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)\) pair of mixed policies.
Practice Exercises
Practice Exercises

6.1 (Mixed security levels/policies - graphical method).
For the two zero-sum matrix games compute the average security levels and all mixed security policies for both players.

\[
A = \begin{bmatrix}
1 & 4 \\
3 & -1
\end{bmatrix}
\] \quad P_1 \text{ choices}

\[
P_2 \text{ choices}
\]

\[
B = \begin{bmatrix}
4 & 0 \\
0 & 2 \\
3 & 1
\end{bmatrix}
\] \quad P_1 \text{ choices}

\[
P_2 \text{ choices}
\]

\[
C = \begin{bmatrix}
1 & 3 & -1 & 2 \\
-3 & -2 & 2 & 1 \\
0 & 2 & -2 & 1
\end{bmatrix}
\] \quad P_1 \text{ choices}

\[
P_2 \text{ choices}
\]

\[
D = \begin{bmatrix}
2 & 1 & 0 & -1 \\
-1 & 3 & 1 & 4
\end{bmatrix}
\] \quad P_1 \text{ choices}

\[
P_2 \text{ choices}
\]

Use the graphical method.

**Hint:** for $3 \times 2$ and $2 \times 3$ games start by computing the average security policy for the player with only two actions.
Practice Exercises

Solution for the Matrix $A$

\[ V_m(A) = \min_y \max_z \quad y_1 z_1 + 4y_1 z_2 + 3y_2 z_1 - y_2 z_2 \]

\[ = \min_y \max\{y_1 + 3y_2, 4y_1 - y_2\} = \min_y \max\{-2y_1 + 3, 5y_1 - 1\} = \frac{13}{7} \]

with the (unique) mixed security policy for $P_1 : y^* := \left[ \frac{4}{7} \quad \frac{3}{7} \right]'$, and

\[ V_m(A) = \max_z \min_y \quad y_1 z_1 + 4y_1 z_2 + 3y_2 z_1 - y_2 z_2 \]

\[ = \max_z \min\{z_1 + 4z_2, 3z_1 - z_2\} = \max_z \min\{-3z_1 + 4, 4z_1 - 1\} = \frac{13}{7} \]

with the (unique) mixed security policy for $P_2 : z^* := \left[ \frac{5}{7} \quad \frac{2}{7} \right]'$. Consequently, this game has a single mixed saddle-point equilibrium \( (y^*, z^*) = \left( \left[ \frac{4}{7} \quad \frac{3}{7} \right]', \left[ \frac{5}{7} \quad \frac{2}{7} \right]' \right) \).
Practice Exercise

Solution for the Matrix $B$

$$V_m(B) = \max_z \min_y 4y_1z_1 + 2y_2z_2 + 3y_3z_1 + y_3z_2$$

$$= \max_z \min\{4z_1, 2z_2, 3z_1 + z_2\} = \max_z \min\{4z_1, 2 - 2z_1, 2z_1 + 1\} = \frac{4}{3}$$

with the sole mixed security policy for $P_2$: $z^* := \left[ \frac{1}{3} \quad \frac{2}{3} \right]'$.

$P_1$ has more than 2 actions:
- cannot use the graphical method to find the mixed security policies for this player.
Practice Exercise

However, from

\[ \{ y \in \mathbb{R}^m : y \geq 0, \ 1' y = 1, \ v^* 1 \geq A' y \} \]

we know that the mixed security policies for \( P_1 \) must satisfy

\[ \frac{4}{3} 1 \geq A' y = \begin{bmatrix} 4y_1 + 3y_3 \\ 2y_2 + y_3 \end{bmatrix} = \begin{bmatrix} y_1 - 3y_2 + 3 \\ -y_1 + y_2 + 1 \end{bmatrix} \]

\[ \Leftrightarrow y_1 \leq -\frac{5}{3} + 3y_2, \ y_1 \geq y_2 - \frac{1}{3}, (y_1 \leq 1 - y_2) \]

which has a single solution \( y^* := \left[ \frac{1}{3} \ 2 \ 0 \right]' \).

Consequently, game has a single mixed saddle-point equilibrium

\[ (y^*, z^*) = \left( \left[ \frac{1}{3} \ 2 \ 0 \right]', \left[ \frac{1}{3} \ 2 \right]' \right) \]
Practice Exercise

Solution for the Matrix $C$

- Row 3 strictly dominates over row 1.
- Column 2 strictly dominates over column 1.

We can therefore reduce the game to

$$C^\dagger := \begin{bmatrix} -2 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

For this matrix

$$V_m(C^\dagger) = \min_y \max \{-2y_1 + 2y_2, 2y_1 - 2y_2, y_1 + y_2\}$$

$$= \min_y \max \{-4y_1 + 2, 4y_1 - 2, 1\} = 1$$

which has multiple minima for $y_1 \in \left[\frac{1}{4}, \frac{3}{4}\right]$. These correspond to the following security policies for $P_1$ in the original game:

$$y^* := \begin{bmatrix} 0 & y_1 & 1 - y_1 \end{bmatrix}' \text{ for any } y_1 \in \left[\frac{1}{4}, \frac{3}{4}\right]$$
Practice Exercise

$P_2$ has more than 2 actions even for the reduced game $C^\dagger$

- cannot use the graphical method to find the mixed security policies for this player.

However, from $\{z \in \mathbb{R}^n : z \geq 0, \ 1'z = 1, \ v^*1 \leq Az\}$

we know that these policies must satisfy

$$1 \leq Az = \begin{bmatrix} -2z_1 + 2z_2 + z_3 \\ 2z_1 - 2z_2 + z_3 \end{bmatrix} = \begin{bmatrix} -3z_1 + z_2 + 1 \\ z_1 - 3z_2 + 1 \end{bmatrix}$$

$$\Leftrightarrow z_2 \geq -3z_1, \ z_2 \leq \frac{1}{3}z_1, (z_2 \leq 1 - z_1)$$

which has a single solution $z_1 = z_2 = 0$, and corresponds to the security policy for $P_2$ in the original game: $z^* := [0 \ 0 \ 0 \ 1]'$.

This game has the family of mixed saddle-point equilibria

$$(y^*, z^*) = ([0 \ y_1 \ 1 - y_1]', [0 \ 0 \ 0 \ 1]'), \ y_1 \in \begin{bmatrix} 1 \\ 4 \\ 3 \\ 4 \end{bmatrix}$$
Practice Exercise

Solution for the Matrix $D$

- Column 2 strictly dominates over column 3. We can therefore reduce the game to

$D^\dagger := \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & 4 \end{bmatrix}$

For this matrix

$V_m(D^\dagger) = \min_y \max \{2y_1 - y_2, y_1 + 3y_2, 4y_2 - y_1\}$

$= \min_y \max \{3y_1 - 1, 3 - 2y_1, 4 - 5y_1\} = \frac{7}{5}$

which has a single minimum for $y_1 = \frac{4}{5}$. This corresponds to the security policy for $P_1$ in the original game: $y^* := \left[\frac{4}{5}, \frac{1}{5}\right]'$.

Since $P_2$ has more than 2 actions even for $D^\dagger$, we cannot use the graphical method to find the mixed security policies.
Practice Exercise

However, from \( \{ z \in \mathbb{R}^n : z \geq 0, \ 1'z = 1, \ v^*1 \leq Az \} \)
we know that these policies must satisfy

\[
\frac{7}{5}1 \leq Az = \begin{bmatrix}
2z_1 + z_2 - z_3 \\
-z_1 + 3z_2 + 4z_3
\end{bmatrix} = \begin{bmatrix}
3z_1 + 2z_2 - 1 \\
-5z_1 + 3z_2 + 4
\end{bmatrix}
\]

\[ \Leftrightarrow z_2 \geq -\frac{3}{2}z_1 + \frac{6}{5}, z_2 \leq -5z_1 + \frac{13}{5}, (z_2 \leq 1 - z_1) \]

which has a single solution \( z_1 = \frac{2}{5}, \ z_2 = \frac{3}{5} \). This corresponds to the security policy for \( P_2 \) in the original game:

\( z^* := \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 0 & 0 \end{bmatrix}' \).

This game has the unique mixed saddle-point equilibrium

\[
(y^*, z^*) = \left( \begin{bmatrix} 4 & 1 \end{bmatrix}', \begin{bmatrix} 2 & 3 & 0 & 0 \end{bmatrix}' \right)
\]
Practice Exercise

6.2 (Mixed security levels/policies - LP method).

For each of the following two zero-sum matrix games compute the average security levels and a mixed security policy.

\[
A = \begin{bmatrix}
3 & 1 \\
2 & 2 \\
1 & 3 \\
\end{bmatrix} \quad P_1 \text{ choices}
\]

\[
B = \begin{bmatrix}
0 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
\end{bmatrix} \quad P_1 \text{ choices}
\]

Solve this problem numerically using MATLAB®.
Practice Exercise

Solution for the matrix $A$

Use the CVX code to compute the mixed value and the security policies for $P_2$ and $P_1$:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

```matlab
A = [3,1;2,2;1,3];

cvx_begin
    variables v z(size(A,2));
    maximize v;
    subject to
        z >= 0;
        sum(z) == 1;
        A*z >= v;
    cvx_end

cvx_begin
    variables v y(size(A,1));
    minimize v;
    subject to
        y >= 0;
        sum(y) == 1;
        A'*y <= v;
    cvx_end
```
Practice Exercise

Code resulted in the mixed value $V_m(A) = 2$ and a saddle-point

$$(y^*, z^*) = \left( \left[ \begin{array}{ccc} 1 & 1 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right]’, \left[ \begin{array}{c} 1 \\ \frac{1}{2} \end{array} \right]’ \right)$$

However, this matrix has multiple saddle-point equilibria and for $y^*$ so you may get any distribution of the form

$$\lambda \left[ 1 - 2\lambda \ 1 \right]’, \ \forall \lambda \in \left[ 0, \frac{1}{2} \right]$$
Practice Exercise

Solution for the matrix $B$

Use the CVX code to compute the mixed value and the security policies for $P_2$ and $P_1$:

\[
A = \begin{bmatrix} 0,1,2,3;1,0,1,2;0,1,0,1;-1,0,1,0 \end{bmatrix}
\]

```matlab
cvx_begin
    variables v z(size(A,2));
    maximize v;
    subject to
        z>=0;
        sum(z)==1;
        A*z>= v;
  cvx_end
```

```matlab
  cvx_begin
    variables v y(size(A,1));
    minimize v;
    subject to
        y>=0;
        sum(y)==1;
        A'*y <= v;
  cvx_end
```
Practice Exercise

Code resulted in the mixed value $V_m(B) = \frac{1}{2}$ and a saddle-point

$$(y^*, z^*) = \left( \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}', \begin{bmatrix} 0 & 0.1858 & \frac{1}{2} & 0.3142 \end{bmatrix}' \right)$$

However, this matrix has multiple saddle-point equilibria and for $z^*$ so you may get any distribution of the form

$$\begin{bmatrix} 0 & \frac{\lambda}{2} & \frac{1}{\lambda} & \frac{1-\lambda}{2} \end{bmatrix}', \forall \lambda \in [0, 1]$$
End of Lecture

06 - Computation of Mixed Saddle-Point Equilibrium Policies

Questions?